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NOTE

Comments on “Filter-Based Stabilization of Spectral Element Methods”

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P. Fischer and J. Mullen [3] have recently proposed a stabilization technique for the solution of the unsteady Navier–Stokes equations with the spectral element method. It is based on interpolations in physical space: Given the values of a function u on a Gauss–Lobatto–Legendre (GLL) mesh with $(N+1)^d$ nodes per element (where d is the space dimension and N is the degree of the polynomial approximation in each direction), in each element one uses the polynomial interpolant to compute u at the N^d nodes–GLL mesh, so that one obtains a new polynomial approximation, the degree of which in each direction is then $N-1$. Combined with a relaxation method, this “filtering” of the highest frequencies is applied at each time step. An important advantage of the technique is that interelement continuity and boundary conditions are preserved. Its efficiency, for high Reynolds number flows, was demonstrated from results of numerical simulations. In the present note we point out that this technique finds a simple interpretation in Legendre spectral space and that this interpretation remains true when a Chebyshev polynomial approximation is used. Although the result is easy to obtain it is not *a priori* obvious, so that to our knowledge this viewpoint has not yet been clearly stated. Moreover, we emphasize the link, only briefly mentioned in the conclusion of [3], to the filtering procedure suggested in [2], and finally we consider the case of the Fourier spectral method, by extension to the trigonometric polynomials.

Without loss of generality we can restrict ourselves to the one-dimensional situation. Let $\Lambda = [-1, 1]$, $P_N(\Lambda)$, $N \in \mathbb{N}^*$, the space of the polynomials of maximum degree N defined on Λ, $\{L_i\}_{i=0}^N$, the set of the $N+1$ Legendre polynomials of degree i, and $\{\xi_i^N\}_{i=0}^N$, the set of the GLL nodes associated with $P_N(\Lambda)$, i.e., those that solve $(1-x^2)L_N'(x) = 0$.

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To every $u \in P_N(\Lambda)$ is associated a polynomial $v \in P_{N-1}(\Lambda)$, such that $v(\xi_i^{N-1}) = u(\xi_i^{N-1}), 0 \le i \le N-1$. We denote $F_N : P_N(\Lambda) \to P_{N-1}(\Lambda)$, the interpolation operator such that $F_N u = v$.

With α a real number such that $0 < \alpha \le 1$, the stabilization operator proposed in [3] gives $F_{N,\alpha} = \alpha F_N + (1-\alpha)\, Id$ where Id is the identity operator. The relaxation parameter α allows us to filter only a fraction of the highest mode.

In fact one can easily check that the operator F_N simply transfers the N-mode value to the $N-2$ one and that this result also holds for the Chebyshev polynomials. More precisely, we have the following:

Given $u \in P_N(\Lambda)$, let $\{\varphi_k\}_{k=0}^N$ be the Legendre or Chebyshev polynomial basis of $P_N(\Lambda)$ and let $\{\hat{u}_k\}_{k=0}^N$ be the components of u in this basis, i.e., $u = \sum_{k=0}^N \hat{u}_k \varphi_k$. Then

$$F_N u = \sum_{k=0}^{N-3} \hat{u}_k \varphi_k + (\hat{u}_{N-2} + \hat{u}_N)\varphi_{N-2} + \hat{u}_{N-1}\varphi_{N-1}. \tag{1}$$

Of course, this result implies

$$F_{N,\alpha} u = \sum_{k=0}^{N-3} \hat{u}_k \varphi_k + (\hat{u}_{N-2} + \alpha \hat{u}_N)\varphi_{N-2} + \hat{u}_{N-1}\varphi_{N-1} + (1-\alpha)\hat{u}_N \varphi_N.$$

Demonstrating Eq. (1) is not difficult. Clearly, $F_N \varphi_k = \varphi_k$ if $k < N$, so that, with $v = F_N u$,

$$\hat{v}_k = \hat{u}_k + \hat{u}_N \hat{p}_k, \quad 0 \le k \le N-1,$$

where $p := F_N \varphi_N = \sum_{k=0}^{N-1} \hat{p}_k \varphi_k$.

It remains to compute the $\hat{p}_k$. Using the N-points-based Gauss–Lobatto quadrature formula, one obtains

$$\int_{-1}^{1} p(x)\varphi_k(x) w(x)\, dx \approx \sum_{i=0}^{N-1} \rho_i^{N-1} p\left(\xi_i^{N-1}\right)\varphi_k\left(\xi_i^{N-1}\right),$$

where $w(x)$ is the appropriate weight function ($w(x) = 1$ in the Legendre case and $w(x) = (1-x^2)^{-0.5}$ in the Chebyshev case) and where the ρ_i^{N-1} are the quadrature coefficients.

But from the definition of F_N, $p(\xi_i^{N-1}) = \varphi_N(\xi_i^{N-1}), 0 \le i \le N-1$. Because the N-points-based Gauss–Lobatto quadrature formula is exact for polynomials of degree n such that $n \le 2N-3$, the orthogonality of the basis $\{\varphi_k\}_{k=0}^N$ means that $\hat{p}_k = 0$, so that $\hat{v}_k = \hat{u}_k$, if $k \le N-3$. Consequently,

$$p = \hat{p}_{N-1}\varphi_{N-1} + \hat{p}_{N-2}\varphi_{N-2}.$$

However, the polynomials φ_{N-1} and φ_N are of opposite parity, whereas φ_{N-2} and φ_N are of same parity. Since it is clear that the operator F_N preserves the parity, we can infer $\hat{p}_{N-1} = 0$. Finally, since F_N does not change the values of p at the endpoints, $\hat{p}_{N-2} = 1$. Consequently, $\hat{v}_{N-2} = \hat{u}_{N-2} + \hat{u}_N$ and $\hat{v}_{N-1} = \hat{u}_{N-1}$.

Note that Eq. (1) may also be obtained in a less direct but quicker way. In fact, from the definition of F_N and the unicity of the Legendre or Chebyshev decomposition, one has simply to check that

$$\varphi_N\left(\xi_i^{N-1}\right) = \varphi_{N-2}\left(\xi_i^{N-1}\right), \quad 0 \leq i \leq N-1. \tag{2}$$

• Legendre case ($\varphi_k = L_k$): Starting from the differential and recurrence relations [1]

$$(1-x^2)L_k' = kL_{k-1} - kxL_k,$$
$$L_{k+1} = \frac{2k+1}{k+1}xL_k - \frac{k}{k+1}L_{k-1},$$

by elimination of xL_k one obtains the new relation

$$(1-x^2)L_k' = \frac{k(k+1)}{2k+1}(L_{k-1} - L_{k+1}).$$

For $k = N-1$, we obtain $L_{N-2}(\xi_i^{N-1}) = L_N(\xi_i^{N-1})$, $0 \leq i \leq N-1$, since from the definition of the GLL points, the ξ_i^{N-1} solve $(1-x^2)L_{N-1}'(x) = 0$.

• Chebyshev case ($\varphi_k = T_k(x) = \cos k(\operatorname{acos} x)$): Because the Chebyshev–Gauss–Lobatto points are explicitly known, i.e., $\xi_i^N = \cos(\pi i/N)$, $0 \leq i \leq N$, the equalities (2) can be trivially obtained as

$$T_N\left(\xi_i^{N-1}\right) = \cos\left(\frac{N\pi i}{N-1}\right) = \cos\left(\pi i + \frac{\pi i}{N-1}\right),$$
$$T_{N-2}\left(\xi_i^{N-1}\right) = \cos\left(\frac{(N-2)\pi i}{N-1}\right) = \cos\left(\pi i - \frac{\pi i}{N-1}\right),$$

so that $T_{N-2}(\xi_i^{N-1}) = T_N(\xi_i^{N-1})$, $0 \leq i \leq N-1$.

At this point, it is of interest to argue that the stabilization technique considered here may be recast in the more general approach introduced by Boyd [2]. Indeed, to filter while preserving the endpoint values it is suggested in [2] to

(a) rewrite $u(x)$ as

$$u(x) = ax + b + U(x),$$

where $U(\pm 1) = 0$ and with a and b determined from $u(\pm 1)$;

(b) express U in a new basis $\{\phi_k\}_{k=0}^{N-2}$, such that $\phi_k(\pm 1) = 0$, of course, the choice of this basis is an important point and using

$$\phi_k = \varphi_{k+2} - \varphi_k$$

is suggested; and

(c) apply the filtering procedure to the spectrum of U in this basis.

First, from the equality

$$\sum_{k=0}^{N} \hat{u}_k \varphi_k(x) = ax + b + \sum_{k=0}^{N-2} \hat{U}_k \phi_k(x),$$

by identification one obtains $\hat{U}_{N-2} = \hat{u}_N$.

Now let us apply a cutoff filter F'_N which simply cancels the $N-2$ -mode of U. Then

$$\begin{aligned} F'_N u &= u - \hat{u}_N(\varphi_N - \varphi_{N-2}) \\ &= \sum_{k=0}^{N-3} \hat{u}_k\varphi_k + (\hat{u}_{N-2} + \hat{u}_N)\varphi_{N-2} + \hat{u}_{N-1}\varphi_{N-1}, \end{aligned}$$

so that $F'_N = F_N$.

Finally it should be stressed that a similar filter-based stabilization technique can be produced for the Fourier spectral method. To this end we focus on the approximation of periodic functions by trigonometric polynomials.

More precisely, let $S_N(\Lambda)$ be the space of the real-valued trigonometric polynomials defined on $\Lambda = [0, 2\pi]$ and of degree k such that $|k| \leq N$. Any u of $S_N(\Lambda)$ may be written as

$$u(x) = \sum_{k=-N}^{N} \hat{u}_k \exp(\mathrm{i}kx), \quad (\mathrm{i}^2 = -1), \tag{3}$$

where $\hat{u}_{-k} = \bar{\hat{u}}_k$ (where $\bar{\hat{u}}_k$ is the complex conjugate of $\hat{u}_k$). Let us define the operator F_N as before and use the regular mesh of $\Lambda : \xi_i^N = \pi i/N, 0 \leq i \leq 2N-1$. Then we have the following:

With $u \in S_N(\Lambda)$ and $\{\hat{u}_k\}_{k=-N}^N$ the Fourier spectrum of u, one has

$$F_N u(x) = \sum_{k=-N+1}^{N-1} \hat{v}_k \exp(\mathrm{i}kx) \tag{4}$$

with

$$\begin{aligned} \hat{v}_k = \hat{u}_k \quad &\text{if } |k| \leq N-3, \quad k = \pm(N-1), \\ \hat{v}_{N-2} &= \hat{u}_{N-2} + \hat{u}_{-N}, \\ \hat{v}_{-N+2} &= \hat{u}_{-N+2} + \hat{u}_N. \end{aligned}$$

Again, because the grid points are explicitly known the result can be trivially obtained, since an elementary calculation yields the two sets of equalities

$$\exp\bigl(\epsilon \mathrm{i} N\xi_i^{N-1}\bigr) = \exp\bigl(-\epsilon\mathrm{i}(N-2)\xi_i^{N-1}\bigr), \quad 0 \leq i \leq 2N-3, \quad \epsilon = \pm 1.$$

However, from a practical point of view, one uses the discrete Fourier transform which associates to the $u(\xi_i^N)$, $0 \leq i \leq 2N-1$, the $\tilde{u}_k$, $0 \leq k \leq N$, such that, with $\tilde{u}_{-k} = \bar{\tilde{u}}_k$ and $\tilde{u}_N$ real,

$$u\bigl(\xi_i^N\bigr) = \sum_{k=-N+1}^{N} \tilde{u}_k \exp\bigl(\mathrm{i}k\xi_i^N\bigr).$$

Here the problem is that to interpolate u at the grid points ξ_i^{N-1}, a real-valued expression like (3) is required. Knowing that $\exp(-\mathrm{i}N\xi_i^N) = \exp(\mathrm{i}N\xi_i^N) = \pm 1$ and that necessarily $\hat{u}_{-N} = \bar{\hat{u}}_N$, it is natural to assume that $\hat{u}_N = \hat{u}_{-N} = \tilde{u}_N/2$.

Then

$$F_N u\left(\xi_i^N\right) = \sum_{k=-N+1}^{N} \tilde{v}_k \exp\left(\mathrm{i}k\xi_i^N\right), \quad 0 \leq i \leq 2N-1,$$

with

$$\tilde{v}_k = \tilde{u}_k \quad \text{if } |k| \leq N-3, \quad k = \pm(N-1),$$
$$\tilde{v}_{\pm(N-2)} = \tilde{u}_{\pm(N-2)} + \frac{\tilde{u}_N}{2},$$
$$\tilde{v}_N = 0.$$

Preliminary numerical tests have shown the expected stabilization effect of this approach in the computation of 3D flows with one homogeneous direction, using spectral Fourier–Legendre elements.

To conclude this note let us mention that the present filter-based stabilization technique appears to be optimal in the following sense: The filtering of the Nth mode while keeping the endpoint values is achieved by modifying only the mode of rank $k \geq k_c$, so that k_c is maximal. Indeed, since the Legendre or Chebyshev polynomials are alternatively odd or even, one has $k_c \leq N-2$. But precisely, in the filter-based stabilization technique, the endpoint values are preserved by modifying only the mode $k_c = N-2$. This may be an important feature in the framework of direct numerical simulations of fluid flows. However, one may notice that, in contrast with usual filters, the filter-based stabilization technique does not imply a dissipation of "energy." Thus, for the Legendre or Chebyshev polynomials, with $\|\cdot\|_w$ for the L^2 weighted norm, one has

$$\|u\|_w^2 - \|F_N u\|_w^2 = \hat{u}_N^2\|\varphi_N\|_w^2 + \hat{u}_{N-2}^2\|\varphi_{N-2}\|_w^2 - (\hat{u}_N + \hat{u}_{N-2})^2\|\varphi_{N-2}\|_w^2,$$

which may be positive or negative. This is particularly obvious for the Chebyshev polynomials, for which $\|\varphi_i\|_w = \frac{\pi}{2}$ for $i > 0$, since then the sign of the difference is simply opposite to the sign of the product $\hat{u}_N\hat{u}_{N-2}$. But as far as we know, such a possible creation/dissipation of energy has never induced any anomalies in the numerical results.

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